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Remarks on the Linear Independence of Integer Translates of Exponential Box Splines

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Following N. Sivakumar (*J. Approx. Theory* **64** (1991), 95–118), we study in this note the problem of the linear independence of the integer translates of an exponential box spline associated with a rational direction set.

The following brief note reacts to the recent interesting paper of N. Sivakumar [S]. As such, we also adhere to the notations used there.

Throughout the discussion, we associate every multiset of n s-dimensional non-trivial real vectors $\Xi = \{\xi_1, ..., \xi_n\}$ with an $s \times n$ matrix whose columns are $\xi_1, ..., \xi_n$, and use the notation Ξ for this associated matrix as well. Given a matrix Ξ and corresponding constants $\lambda := \{\lambda_\xi\}_{\xi \in \Xi} \subset \mathbb{C}$, the exponential box spline $B_{\Xi,\lambda}$ is defined $[\mathrm{R}_1]$, as the distribution whose Fourier transform is

$$\hat{B}_{\Xi,\lambda}(x) = \prod_{\xi \in \Xi} \int_0^1 e^{(\lambda_\xi - i\xi \cdot x)t}\, dt. \tag{1}$$

We refer to [S] and the references therein for further discussion of exponential box splines. Here, we are interested solely in dependence relations for the integer translates of $B_{\Xi,\lambda}$. Precisely, defining

$$K(B_{\Xi,\lambda}) := \left\{ a\colon \mathbb{Z}^s \to \mathbb{C}\colon \sum_{j \in \mathbb{Z}^s} a(j)\, B_{\Xi,\lambda}(\cdot - j) = 0 \right\}, \tag{2}$$

we wish to know when $K(B_{\Xi,\lambda})$ is trivial or at least finite-dimensional. We note that the sum in (2) is always well defined, since $B_{\Xi,\lambda}$ is compactly supported.

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Whenever $K(B_{\Xi,\lambda}) = \{0\}$, the integer translates of $B_{\Xi,\lambda}$ are linearly independent. This question of linear independence has received major attention in box spline theory (see the discussion in [S]), with the analysis being focused, however, on the integer case, i.e., when Ξ is an integral matrix. It seems that only [JS] and [S] (and also the example in the last section of [CR]) provide results concerning *rational* matrices Ξ. Furthermore, the examples in [JS] indicate that in the rational case there probably exists no satisfactory characterization for the linear independence of the integer translates.

Interesting sufficient conditions for $K(B_{\Xi,\lambda})$ being trivial or finite-dimensional have been obtained in [S]. Our aim here is to derive slightly more general results, and with the aid of a different approach: while the proofs in [S] (as well as in [JS]; see also the approach in [J]) proceed by an involved induction on s and n, and require as a preparation a certain transformation to be applied to Ξ, here we make use of observations and arguments from the theory of the integer case. In addition, this approach links the two main results of [S].

We start by recalling from [S] the notion of *extendibility*:

DEFINITION. Let $Y \subset \mathbb{Q}^s$ be a linearly independent set of $1 \leqslant k \leqslant s$ vectors. We say that Y is *extendible* (or possesses property E) if there is a matrix $X_{s\times s}$ with an integral inverse whose first k columns constitute Y. Also, for an arbitrary $s \times n$ matrix Ξ, we say that Ξ is *fully extendible* if every linearly independent subset Y of Ξ is extendible.

Note that Ξ is fully extendible if and only if every basis Y of the column span of Ξ is extendible.

As in [JS] and [S], we follow [R_2] and introduce, for a compactly supported distribution ψ, the set

$$N(\psi) = \{\theta \in \mathbb{C}^s: \hat{\psi}(\theta + 2\pi\alpha) = 0, \forall \alpha \in \mathbb{Z}^s\}. \tag{3}$$

(4) THEOREM (Sivakumar [S]). *Let $B_{\Xi,\lambda}$ be an exponential box spline with a rational set of directions. Then the integer translates of $B_{\Xi,\lambda}$ are linearly independent if the following two conditions hold*:

(a) *Ξ is fully extendible*;

(b) *$\hat{B}_{\Xi,\lambda}$ vanishes nowhere on the set $-i\Theta_\lambda(\Xi)$, with*

$$\Theta_\lambda(\Xi) := \{\phi \in \mathbb{C}^s: \operatorname{span}\{\xi \in \Xi: \xi \cdot \phi = \lambda_\xi\} = \operatorname{span} \Xi\}. \tag{5}$$

Proof. By [R_2, Theorem 1.1], $K(B_{\Xi,\lambda}) = \{0\}$ if and only if $N(B_{\Xi,\lambda}) = \varnothing$. Assume that $\theta \in \mathbb{C}^s$. To show that $\theta \notin N(B_{\Xi,\lambda})$, we need to find $\alpha \in \mathbb{Z}^s$ such that $\hat{B}_{\Xi,\lambda}(\theta + 2\pi\alpha) \neq 0$. The argument for that follows closely the proof of Theorem 1.4 in [R_2].

For each $\xi \in \Xi$ we set

$$v_\xi := \frac{i\lambda_\xi + \theta \cdot \xi}{2\pi}. \tag{6}$$

In view of (1), the desired $\alpha \in \mathbb{Z}^s$ should satisfy

$$v_\xi + \alpha \cdot \xi \notin \mathbb{Z}\backslash 0, \qquad \forall \xi \in \Xi. \tag{7}$$

Let Y be a maximally linearly independent subset of $\{\xi \in \Xi\colon v_\xi \in \mathbb{Z}\}$ (the possibility $Y = \varnothing$ is not excluded). By condition (a), Y is extendible to a matrix with integral inverse, and therefore the system

$$v_y + ? \cdot y = 0, \qquad y \in Y,$$

admits an integral solution $? = \alpha_1$. We now replace each v_ξ ($\xi \in \Xi$) by $v_\xi^1 := v_\xi + \alpha_1 \cdot \xi$. Note that $v_y^1 = 0$ for every $y \in Y$. We need to overcome the difficulty occurring when some of the v_ξ^1's are non-zero integers. We first show that this is impossible for $\xi \in \operatorname{span} Y$.

Let $\xi \in (\operatorname{span} Y) \cap \Xi$, $\xi = \sum_{y \in Y} \beta_y y$. Choose $\phi \in \Theta_\lambda(\Xi)$ such that $\lambda_y - \phi \cdot y = 0$ for every $y \in Y$. Denoting $\theta' := \theta + 2\pi\alpha_1$, we have $2\pi v_\xi^1 = i\lambda_\xi + \theta' \cdot \xi$ for every $\xi \in \Xi$, hence for every $y \in Y$, $\theta' \cdot y = -i\lambda_y$ (since $v_y^1 = 0$); therefore

$$\begin{aligned} v_\xi^1 &= \frac{i\lambda_\xi + \theta' \cdot \sum_{y \in Y} \beta_y y}{2\pi} \\ &= \frac{i\lambda_\xi + \sum_{y \in Y} \beta_y \theta' \cdot y}{2\pi} \\ &= \frac{i\lambda_\xi - i\sum_{y \in Y} \beta_y \lambda_y}{2\pi} \\ &= \frac{i\lambda_\xi - i\sum_{y \in Y} \beta_y \phi \cdot y}{2\pi} \\ &= \frac{i(\lambda_\xi - \phi \cdot \xi)}{2\pi} \notin \mathbb{Z}\backslash\{0\}, \end{aligned} \tag{8}$$

where in the last step we have used condition (b) (if $i(\lambda_\xi - \phi \cdot \xi)/2\pi \in \mathbb{Z}\backslash\{0\}$, then, by (1), $\hat{B}_{\xi, \lambda_\xi}(-i\phi) = 0$, *a fortiori* $\hat{B}_{\Xi, \lambda}(-i\phi) = 0$).

Let Y_1' be the set of all $\xi \in \Xi$ that satisfy $v_\xi^1 \in \mathbb{Z}\backslash\{0\}$. If $Y_1' \neq \varnothing$, then, with $\xi \in Y_1'$ chosen arbitrarily, we conclude from the previous argument that $Y_1 := Y \cup \{\xi\}$ is still linearly independent. Replacing Y by Y_1, we repeat the previous step: we find $\alpha_2 \in \mathbb{Z}^s$ that satisfies $v_y^1 + \alpha_2 \cdot y = 0$ for every $y \in Y_1$, then define $v_\xi^2 := v_\xi^1 + \alpha_2 \cdot \xi$ for every $\xi \in \Xi$, and conclude that if $Y_2' := \{\xi \in \Xi\colon v_\xi^2 \in \mathbb{Z}\backslash\{0\}\} \neq \varnothing$, then the set $Y_1 \cup \{\xi\}$ is linearly independent, with $\{\xi\}$ being arbitrarily chosen from Y_2'. After finitely many (say, j) steps we must get $Y_j' = \varnothing$, such that all v_ξ^j are not in $\mathbb{Z}\backslash\{0\}$. Since

$$v_\xi^j = v_\xi + \left(\sum_{k=1}^{j} \alpha_k\right) \cdot \xi,$$

we conclude (in view of (7)) that $\alpha := \sum_{k=1}^{j} \alpha_k$ is the required integer. ∎

Next we consider the question of the finite dimensionality of $K(B_{\Xi,\lambda})$. The results below will show that in essence this question is not harder than the linear independence one. It is the lack of a good characterization of the latter case that prevents us from establishing a good characterization for the finite dimensionality problem.

We first recall the following fact. The "only if" implication of it follows from [R_2, Theorem 1.1], while the "if" implication has been proved in [DJM, Theorem 2.1].

(9) RESULT. *Let ψ be a compactly supported distribution. Then $K(\psi)$ is finite-dimensional if and only if $N(\psi)/2\pi\mathbb{Z}^s$ is finite.*

The following extends a result which has been (implicitly) proved in [DM_1] for polynomial box splines with an integral set of directions. For *exponential* box splines with an integral set of directions Ξ, a weaker form of this result follows from [DM_2, Theorem 7.2]; see also Theorem 4.2 of [S]. Note that we do *not* assume Ξ in the theorem to be spanning.

(10) THEOREM. *Let $B_{\Xi,\lambda}$ be an exponential box spline with a rational set of directions. Then $K(B_{\Xi,\lambda})$ is infinite-dimensional if and only if $K(B_{Y,\lambda_Y})$ (with $\lambda_Y := \{\lambda_y\}_{y\in Y}$) is non-trivial for some subset $Y \subset \Xi$ of rank $< s$.*

Proof. Suppose first that for some non-spanning $Y \subset \Xi$ and $\theta \in \mathbb{C}^s$, $\theta \in N(B_{Y,\lambda_Y})$. Since $\hat{B}_{Y,\lambda_Y}$ is constant along directions orthogonal to span Y, it follows that $\theta + x \in N(B_{Y,\lambda_Y})$ for every $x \perp$ span Y, hence $N(B_{Y,\lambda_Y})/2\pi\mathbb{Z}^s$ is infinite, and so is $N(B_{\Xi,\lambda})/2\pi\mathbb{Z}^s$, since $\hat{B}_{Y,\lambda_Y}$ divides $\hat{B}_{\Xi,\lambda}$. We conclude from (9) Result that $K(B_{\Xi,\lambda})$ is infinite-dimensional.

Conversely, assume that $K(B_{Y,\lambda_Y})$ is trivial for every non-spanning $Y \subset \Xi$. Choose a positive integer m such that $m\xi \in \mathbb{Z}^s$ for every $\xi \in \Xi$. Let $\theta \in N(B_{\Xi,\lambda})$. By our assumption, $K(B_{Y,\lambda_Y}) = \{0\}$ for every non-spanning Y, or equivalently [R_2], $N(B_{Y,\lambda_Y}) = \varnothing$ for such Y. Therefore, there must exist *s linearly independent* elements $X = \{\xi_1, ..., \xi_s\} \subset \Xi$ and corresponding $\{\alpha_1, ..., \alpha_s\} \subset \mathbb{Z}^s$ such that

$$\hat{B}_{\xi_j,\lambda_{\xi_j}}(\theta + 2\pi\alpha_j) = 0, \qquad j = 1, ..., s,$$

which implies (1), that

$$\lambda_{\xi_j} - i\xi_j \cdot (\theta + 2\pi\alpha_j) \in 2\pi i\mathbb{Z}, \tag{11}$$

and hence also, with $y_j := m\xi_j$ and $\mu_j := m\lambda_{\xi_j}$, that

$$\mu_j - iy_j \cdot (\theta + 2\pi\alpha_j) \in 2\pi i\mathbb{Z}. \tag{12}$$

Further, $y_j \cdot \alpha_j \in \mathbb{Z}$, so we finally obtain

$$\mu_j - iy_j \cdot \theta \in 2\pi i\mathbb{Z}, \qquad j = 1, ..., s. \tag{13}$$

It is not hard to prove that (13) admits only finitely many solutions $\bmod 2\pi\mathbb{Z}^s$. (In fact, [DM_2, Lemma 6.1] and [BR, Lemma 5.1] show that there are exactly $\det(mX)$ solutions $\bmod 2\pi\mathbb{Z}^s$ for (13), regardless of the choice of the μ's and subjected only to the restriction that $(mX) \subset \mathbb{Z}^s$ is a basis for $\mathbb{R}^s$.) Thus, since the number of bases for $\mathbb{R}^s$ selected from Ξ is finite, we obtain that necessarily $N(B_{\Xi,\lambda})/2\pi\mathbb{Z}^s$ is finite, and our claim follows from (9) Result. ∎

Note that the argument in the first part of the proof is valid for a more general setting: if $\sigma = \psi * \tau$, all being compactly supported and ψ is a measure supported on a proper linear manifold of $\mathbb{R}^s$, then $K(\sigma)$ is finite-dimensional only if $K(\psi) = \{0\}$.

Combining (4) Theorem and (10) Theorem, we recover the second main result of [S]:

(14) Corollary (Sivakumar [S]). *Let $B_{\Xi,\lambda}$ be an exponential box spline with a rational set of directions Ξ. Then $K(B_{\Xi,\lambda})$ is finite-dimensional if for every non-spanning subset $Y \subset \Xi$, the exponential box spline B_{Y,λ_Y} satisfies conditions* (a) *and* (b) *of* (4) *Theorem.*

The proof is now evident: if $Y \subset \Xi$ satisfies conditions (a) and (b), then by (4) Theorem, $K(B_{Y,\lambda_Y}) = \{0\}$. This being true for every non-spanning $Y \subset \Xi$, (10) Theorem implies that $K(B_{\Xi,\lambda})$ is finite-dimensional.

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